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Krichever–Novikov Operator Formalism and Sewing Conformal Fields

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The Krichever–Novikov (KN) operator formalism for a nonchiral bosonic system is extended to higher-genus Riemann surfaces with $K+2$ punctures. This completes the formulation of the KN operator formalism and preparation for describing bosonic string scattering amplitudes. We elaborate further on the sewing prescription.

1. INTRODUCTION

There are two main ways of studying string perturbative theories, Polyakov's (1981*a*,*b*) functional integral formulation and the operator approach (Friedan *et al.*, 1986; Ishiashi *et al.*, 1986; Vafa, 1987*a*; Alvarez-Gaumé *et al.*, 1987, 1988, 1988/89; Witten, 1988; Cheng, 1989*a*). The former has been studied intensively in the last decade and much understanding of string perturbative theory has also been obtained. Recently, however, a number of ambiguities have made the analysis of higher-order string loops, and of questions related to the finiteness of string theory, far more difficult than originally envisaged.

The operator formalism has been gaining momentum over recent years thanks to its conciseness and economy, based on the universal Grassmanian manifold (UGM) as a mathematical foundation. Unfortunately, a conformal field theory formulated in this way over a nontrivial topology looks rather involved. What is more, the operator formalism so obtained is not well-defined and self-contained, since the conserved charges are obtained by using symmetries of the path integral. That is to say, they depend on the path integral formulation of a quantum field theory.

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Based on the work of Krichever and Novikov (1987) (KN), for a nonchiral bosonic system we have given a new formulation called a KN operator formalism (Bonora *et al.*, 1988; Cheng, 1989*b*, 1990). Here we will extend it to higher-genus Riemann surfaces with $K+2$ punctures.

The paper is organized as follows. In Section 2 we review the main results of the KN operator formalism developed by us (Cheng, 1989*b*, 1990). In Section 3 we construct a Virasoro-like algebra and a Hilbert space for the system. Section 4 contains our new main results, a well-defined prescription for sewing conformal field theories on two Riemann surfaces and an operator formalism of correlation functions of conformal fields. In Section 5, we remark on the sewing definition of conformal field theories. Finally, Section 6 points out some open questions that are discussed elsewhere.

2. A KN OPERATOR FORMALISM FOR A BOSONIC STRING

We study a nonchiral bosonic system on a genus-g Riemann surface Σ with action

$$S=\int \partial\bar{\partial}X \tag{1}$$

In local coordinates the equation of motion is

$$\partial\bar{\partial}X=0 \tag{2}$$

Suppose $X(z,\bar{z})$ is defined and nonsingular on $\Sigma-D_+-D_-$ with local coordinates $z_\pm$ vanishing at $P_\pm$, the $D_\pm$ being neighborhoods of $P_\pm$. The general solution to the field equation can then be decomposed into two pieces,

$$X(z,\bar{z})=X^h(z)+X^a(\bar{z}) \tag{3}$$

where $X^h(z)$ is holomorphic and $X^a(\bar{z})$ antiholomorphic. These functions can be expanded in the KN basis. The conjugate momenta of $P^h(z)$ and $P^a(\bar{z})$ are holomorphic and antiholomorphic one-forms, respectively.

Let us introduce the Poisson brackets

$$[X^h(p),P^h(p')]=2\pi\Delta^h_\tau(p,p')$$

$$\Delta^h_\tau(p,p')\equiv\frac{1}{2\pi i}\sum_j A_j(p)\omega_j(p'),\qquad p,p'\in C_\tau \tag{4}$$

If the background space in which the bosonic system lives is D-dimensional, the above brackets are directly generalized to

$$[X^{\mu h}(p), X^{\nu h}(p')] = 2\pi\eta^{\mu\nu}\Delta_\tau^h(p,p'),$$
$$p, p' \in C_\tau, \qquad \mu, \nu = 1, 2, \ldots, D \tag{5}$$

Here C_τ is a closed contour on Σ called a level line, defined as follows:

$$C_\tau = \left\{ Q \in \Sigma,\ R_e\left(\int_{Q_0}^{Q} \omega_{g/2}\right) = \tau \right\} \tag{6}$$

As $\tau \to \pm\infty$, C_τ becomes small circles around $P_\pm$.

According to the canonical quantization, we have the canonical commutators

$$[X_i^{\mu h}, P_j^{\nu h}] = \eta^{\mu\nu}\delta_{ij}, \qquad [X_i^{\mu h}, X_j^{\nu h}] = [P_i^{\mu h}, P_j^{\nu h}] = 0 \tag{7}$$

For the $\bar{z}$-dependent piece, we obtain similar commutation relations

$$[X_i^{\mu a}, P_j^{\nu a}] = \eta^{\mu\nu}\delta_{ij}, \qquad [X_i^{\mu a}, X_j^{\nu a}] = [P_i^{\mu a}, P_j^{\nu a}] = 0 \tag{8}$$

In addition, we have four more commutators between z-dependent and $\bar{z}$-dependent objects:

$$[X_i^{\mu h}, X_j^{\nu a}] = [X_i^{\mu h}, P_j^{\nu a}] = 0, \qquad [P_i^{\mu h}, P_j^{\nu a}] = [P_i^{\mu h}, X_j^{\nu a}] = 0 \tag{9}$$

To summarize, equations (7)–(9) are a complete operator algebra for the bosonic system. The algebra services to define a Fock space. This can be carried out as done in Cheng (1989*b*, 1990). As a result, the full Fock space for the system is the tensor product of $\mathscr{H}$ and $\bar{\mathscr{H}}$, since the two sets of operator algebras commute as mentioned above,

$$H = \mathscr{H} \otimes \bar{\mathscr{H}} \tag{10}$$

In this way we have completed the element of an operator formalism for a nonchiral bosonic system on a higher-genus Riemann surface.

It is necessary to develop the above formalism further in order to calculate string scattering amplitudes. In fact, two points of $P_\pm$ may be viewed as two punctures on a Riemann surface. Calculating scattering amplitudes involves Riemann surfaces with two or more punctures. So we need to construct the KN operator formalism on such Riemann surfaces. Our basic idea is that the conformal field theory on a Riemann surface $M \infty N$ can be constructed according to a well-defined prescription for a sewing construction of conformal field theories if conformal field theories on surfaces M and N are naturally defined.

3. VIRASORO-LIKE ALGEBRA AND HILBERT SPACE

It is well known that the field $X(z, \bar{z})$ itself is not a conformal field (Friedan *et al.*, 1986). However, any finite-order derivative of it multiplied by e^{ikz} and their linear combinations are conformal. By convention, we denote them as $\varphi_i(z, \bar{z})$.

The energy-momentum tensor T_{zz} ($T_{\bar{z}\bar{z}}$) is of great importance in conformal field theories. It is defined by (Belavin *et al.*, 1984)

$$\begin{aligned} T_{zz} &= -\tfrac{1}{2} \lim_{z\to w}\left(\partial_z X\, \partial_w X + D\frac{1}{(z-w)^2}\right) \\ T_{\bar{z}\bar{z}} &= -\tfrac{1}{2} \lim_{\bar{z}\to \bar{w}}\left(\partial_{\bar{z}} X\, \partial_{\bar{w}} X + D\frac{1}{(\bar{z}-\bar{w})^2}\right) \end{aligned} \tag{11}$$

Due to the term $D/(z-w)^2$, T_{zz} ($T_{\bar{z}\bar{z}}$) is not really a rank-two tensor. Under a holomorphic coordinate transformation $z \to w = w(z)$, it is transformed as

$$T'_{ww} = \left(\frac{\partial z}{\partial w}\right)^2 T_{zz} + \frac{1}{12} D \left\{ \frac{\partial^3 z/\partial w^3}{\partial z/\partial w} - \frac{3}{2}\frac{(\partial^2 z/\partial w^2)^2}{(\partial z/\partial w)^2} \right\} \tag{12}$$

If the covering of Σ is part of a projective structure, i.e., the transition functions are in $Sl(2, c)$, the Schwarzian derivative vanishes. Thus, for a given meromorphic vector field ξ on Σ which is holomorphic outside of $P_\pm$, one has the Virasoro generator

$$L(\xi) = \pm \oint_{C_\pm} \xi T \tag{13}$$

and T here is a two-form. The Virasoro algebra (Alberty *et al.*, 1988; Huang and Zhao, 1989; Liu and Ni, 1989; Konisi *et al.*, 1989) with central term is

$$[L(\xi), L(\eta)] = L[[\xi, \eta]] \pm \frac{D}{24\pi i} \oint_{C_\pm} dz_\pm\, \xi(z_\pm) \frac{d^3\eta(z_\pm)}{dz_\pm^3} \tag{14}$$

with $[\xi, \eta]$ the Lie derivative of the vector fields.

We introduce a basis of two-forms Ω_i dual to the vector fields e_i such that the following relation holds:

$$\pm \oint_{C_\pm} e_i \Omega_j = \delta_{ij} \tag{15}$$

and we expand the stress-energy tensor T in terms of Ω_j,

$$T = L_j \Omega_j \tag{16}$$

It is easy to check that

$$L_i = L[e_i] \tag{17}$$

$$[L_i, L_j] = \sum_{s=-3g/2}^{3g/2} C_{ij}^s L_{i+j-s} + D\chi(e_i, e_j) \tag{18}$$

where the coefficients C_{ij}^s and the central term $\chi(e_i, e_j)$ are given as follows:

$$C_{ij}^s = \pm \frac{1}{2\pi i} \oint_{C_\pm} [e_i, e_j]\Omega_{i+j-s} \tag{19}$$

$$\chi(\xi, \eta) = \pm \frac{1}{24\pi i} \oint_{C_\pm} dz_\pm\, \xi(z_\pm) \frac{d^3\eta(z_\pm)}{dz_\pm^3} \tag{20}$$

In the same way one can show that the operator $\bar{L}_i$ corresponding to the $\bar{z}$-dependent $\bar{T}$ also satisfies the Virasoro-like algebra

$$[\bar{L}_i, \bar{L}_j] = \sum_{s=-3g/2}^{3g/2} \bar{C}_{ij}^s \bar{L}_{i+j-s} + D\bar{\chi}(e_i, e_j) \tag{21}$$

and

$$[L_i, \bar{L}_j] = 0 \tag{22}$$

where $\bar{C}_{ij}^s$ and $\bar{\chi}(e_i, e_j)$ are complex conjugates of C_{ij}^s and $\chi(e_i, e_j)$, respectively.

Let $\{\varphi_i(z, \bar{z})\}$ be a complete set of primary fields (Bonora *et al.*, 1989) for the conformal field theory under consideration. They transform in the simplest way,

$$\varphi_i(z, \bar{z}) \to \left(\frac{\partial z'}{\partial z}\right)^{\Delta_i} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{\Delta}_i} \varphi_i(z', \bar{z}') \tag{23}$$

In order to coincide with the transformation property of the fields under diffeomorphisms, the singularities of the primary fields should satisfy the conditions

$$T_{zz}\varphi_i(z', \bar{z}') = \frac{\Delta_i}{(z-z')^2} \varphi_i(z', \bar{z}') + \frac{1}{z-z'} \partial_{z'}\varphi_i(z', \bar{z}') + \text{finite} \tag{24}$$

$$T_{\bar{z}\bar{z}}\varphi_i(z', \bar{z}') = \frac{\bar{\Delta}_i}{(\bar{z}-\bar{z}')^2} \varphi_i(z', \bar{z}') + \frac{1}{\bar{z}-\bar{z}'} \partial_{\bar{z}'}\varphi_i(z', \bar{z}') + \text{finite} \tag{25}$$

They are equivalent to the requirements

$$L_j\varphi_i = \bar{L}_j\varphi_i = 0, \qquad j \geq \tfrac{3}{2}g \tag{26}$$

$$L_0\varphi_i = \Delta_i\varphi_i, \qquad \bar{L}_0\varphi_i = \bar{\Delta}_i\varphi_i \tag{27}$$

which are the generalization of the conditions imposed on the local fields in the Riemann sphere case. Since the primary fields are complete an arbitrary field in the theory can be expanded as a linear combination of operators of the form

$$\mathscr{L}_{-n}\varphi_i = L_{-n_1} \ldots L_{-n_j}\bar{L}_{-\bar{n}_1} \ldots \bar{L}_{-\bar{n}_j}\varphi_i \tag{28}$$

where

$$n_1 \geq n_2 \cdots n_j \geq \tfrac{3}{2}g, \qquad \bar{n}_1 \geq \bar{n}_2 \geq \cdots \bar{n}_j \geq \tfrac{3}{2}g$$

Further, we define physical states corresponding to the operators $\mathscr{L}_{-n}\varphi_i$ by

$$|\mathscr{L}_{-n}\varphi_i\rangle = \mathscr{L}_{-n}\varphi_i|O\rangle_\Sigma \otimes |\bar{O}\rangle_\Sigma \tag{29}$$

In this way the Hilbert space is constructed completely for the states created by the primary fields and their descendants.

4. SEWING CONFORMAL FIELDS AND THE KN OPERATOR FORMALISM

That sewing two Riemann surfaces M and N with punctures yields a new Riemann surface $M\infty N$ suggests a very intuitive method for constructing a genus-g surface out of two surfaces of lower genus; this is a purely mathematical question that has been solved by mathematicians. Simply speaking, one sews two Riemann surfaces M and N, and close coordinates z_P and z_Q which identify neighborhoods of P and Q with open discs in the complex plane. One thus constructs a new surface $\Sigma = M\infty N$ of genus $g = g_1 + g_2$ by identifying points on M and N which satisfy $z_P = q/z_Q$. Our main concern here is to give a well-defined prescription for sewing conformal field theories (Vafa, 1987*b*; Sonoda, 1988*a*; Leclair, 1988) under sewing Riemann surfaces. To do this, we extend the ideas in Cheng (1989*b*, 1990) and Sonoda (1988*a*).

Let $\theta_1 \ldots \theta_K$ be local fields on M outside the discs $\{|z_\pm| < 1/\gamma\}$ around the points $P_\pm$, and let $\theta_{K+1} \ldots \theta_{K+L}$ be local fields on N outside the discs $\{|w_\pm| < 1/\gamma\}$ around $Q_\pm$. Their correlation function on $M\infty N$ outside P_- and Q_+ is defined as follows:

$$\langle \theta_1 \ldots \theta_K \theta_{K+1} \ldots \theta_{K+L}\rangle_{m\infty N} \equiv \sum_{imn} \langle \theta_1 \ldots \theta_K \mathscr{L}_{-n}\varphi_i(P_+)\rangle_M \times \mu_{imn}^{-1} \langle \mathscr{L}_{-n}\tilde{\varphi}_i(Q_-)\theta_{K+1} \ldots \theta_{K+L}\rangle_N \tag{30}$$

where i runs over all primary fields, m and n their descendants, and μ_{imn} is a matrix defined on a sphere S^2

$$\delta_{ij}\mu_{imn} \equiv \langle \mathscr{L}_{-m}\varphi_i(w=0)\mathscr{L}_{-n}\tilde{\varphi}_i(z=0)\rangle_{S^2} \tag{31}$$

We suppose that μ_i is invertible.

Similarly, one could also do sewing on a given single Riemann surface M by removing two points P_1 and P_2 from the surface and identifying discs as above, and obtain a new surface denoted by $M8=\Sigma$ with a genus of $g+1$. In this case, the sewn correlation function is of the form

$$\langle \theta_1 \ldots \theta_k \rangle_{M8} = \sum_{imn} \langle \mathscr{L}_{-m}\tilde{\varphi}_i(P_2)\theta_1 \ldots \theta_k \mathscr{L}_{-n}\varphi_i(P_1)\rangle_M \mu^{-1}_{imn} \tag{32}$$

Note that the $\mathscr{L}_{-m}\tilde{\varphi}$ are the conjugate fields corresponding to the $\mathscr{L}_{-m}\varphi_i$ in the above equations.

Let us assume that the metrics around neighborhoods of $P_\pm$ on M are given by

$$g_{z_\pm z_\pm} = 1/|z_\pm|^2 \tag{33}$$

in the discs $\{|z_\pm| < \gamma\}$. We consider a correlation function of local fields

$$\mathscr{C} \equiv \langle \theta_1(z_1, \bar{z}_1) \ldots \theta_K(z_k, \bar{z}_k)\rangle_M \tag{34}$$

where $\gamma > |z_1| > |z_2| \cdots |z_K| > 0$. We will show that we can calculate the correlation function as a matrix element of operators between two states, namely

$$\mathscr{C} \equiv \langle \Omega | \hat{\theta}_1(z_1, \bar{z}_1) \ldots \hat{\theta}_k(z_k, \bar{z}_k) | 1 \rangle \tag{35}$$

The state $|1\rangle$ is fixed and the state $|\Omega\rangle$ uniquely determined by the complex structure of M and the choice of z up to its normalization, and $\hat{\theta}(z, \bar{z})$ represents an operator associated with a local field $\theta(z, \bar{z})$.

We consider a Riemann surface M with $K+2$ punctures by sewing a Riemann surface M' with 2 punctures $P_\pm$ and K Riemann spheres each with 3 punctures (Figure 1).

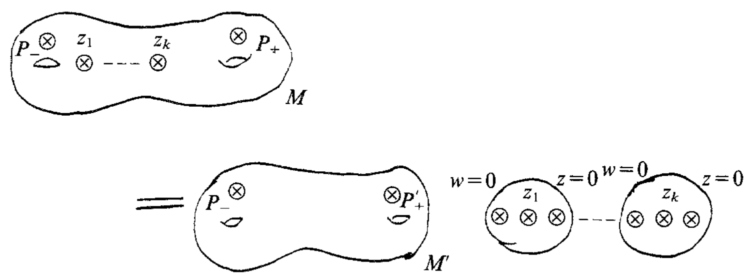


Fig. 1. A Riemann surface with $K+2$ punctures.

Then the sewing prescription (30) gives $\mathscr{C}$ in the following form:

$$\mathscr{C}=\sum_{i_1m_1n_1\ldots i_km_kn_k}\langle\mathscr{L}_{-m_1}\varphi_{i_1}(P'_+)\rangle_{M'}$$
$$\times\,\mu^{-1}_{i_1m_1n_1}\langle\mathscr{L}_{-n_1}\varphi_{i_1}(w=0)\theta_1(z,\bar{z}_1)\mathscr{L}_{-n_2}\tilde{\varphi}_{i_2}(z=0)\rangle_{s^2}$$
$$\times\cdots\times\mu^{-1}_{i_km_kn_k}\langle\mathscr{L}_{-n_k}\tilde{\varphi}_{i_k}(w=0)\theta_k(z_k,\bar{z}_k)\rangle_{s^2} \qquad (36)$$

The two punctures $P_\pm$ on M are the same as the rightmost one on the kth sphere S^2 and P_- on M, respectively (see also Figure 1).

The matrix element of an operator $\hat{\theta}(z,\bar{z})$ acting on the Hilbert space $H=\mathscr{H}\otimes\bar{\mathscr{H}}$ is defined as

$$\langle\mathscr{L}_{-m}\varphi_i|\hat{\theta}(z,\bar{z})|\mathscr{L}_{-n}\tilde{\varphi}_j\rangle\equiv\langle\mathscr{L}_{-m}\varphi_i(w=0)\theta(z,\bar{z})\mathscr{L}_{-n}\tilde{\varphi}_j(z=0)\rangle_{s^2} \qquad (37)$$

The above correlation function may thus be rewritten as follows:

$$\mathscr{C}=\sum_{m_1i_1n_1\ldots m_ki_kn_k}\langle\mathscr{L}_{-m_1}\varphi_{i_1}(P'_+)\rangle_{M'}\mu^{-1}_{i_1m_1n_1}\langle\mathscr{L}_{-n_1}\varphi_{i_1}|$$
$$\times\,\hat{\theta}_1(z_1,\bar{z}_1)|\mathscr{L}_{-n_2}\tilde{\varphi}_{i_2}\rangle\mu^{-1}_{i_2m_2n_2}\langle\mathscr{L}_{-n_2}\tilde{\varphi}_{i_2}|$$
$$\times\cdots\times\hat{\theta}_k(z_k,\bar{z}_k)|1\rangle \qquad (38)$$

It is easily seen that the operator

$$\hat{1}\equiv\sum_{i,m,n}|\mathscr{L}_{-m}\varphi_i\rangle\mu^{-1}_{imn}\langle\mathscr{L}_{-n}\tilde{\varphi}_i| \qquad (39)$$

is the identity one. This comes from the definition of inner products of states. Defining again a state as

$$\langle\Omega|\equiv\sum_{i_1m_1n_1}\langle\mathscr{L}_{-m_1}\varphi_{i_1}(P'_+)\rangle_{M'}\mu^{-1}_{i_1m_1n_1}\langle\mathscr{L}_{-n_1}\hat{\varphi}_{i_1}| \qquad (40)$$

one obtains equation (35), as expected, where

$$\langle\mathscr{L}_{-m_1}\varphi_{i_1}(P'_+)\rangle_{M'}={}_{M'}\langle\bar{o}|\otimes_{M'}\langle O|\mathscr{L}_{-m_1}\hat{\varphi}_{i_1}(P'_+)|0\rangle_{M'}\otimes|\bar{O}\rangle_{M'} \qquad (41)$$

Thus, one has a KN operator formulation for conformal field theory on Riemann surfaces with $K+2$ punctures.

Once we have the KN operator formalism, we can apply it to computing scattering amplitudes of closed bosonic strings. It may give new insight into string perturbative theories. Of course, there are questions which remain open for the operator formalism.

5. REMARK ON THE PRESCRIPTION

The two consistency conditions for the sewing prescription (30) have been checked (Sonoda, 1988*a*). One is the smoothness of the correlation function. When a local field θ is located on the annulus $\{1/r<|z|<r\}$, it can also be thought of as being located on the annulus $\{1/r<|w|<r\}$. Thus, there are two ways of defining the correlation function of θ using (30). It is clear that the two must agree, and it has been shown that they do. Another question is the physical meaning of the energy-momentum tensor. One must make sure that T_{zz} defined on $M\infty N$ by equation (30) also generates a deformation of the complex structure of $M\infty N$. It has been verified that T_{zz} plays the role of a generator of deformations for the conformal theory of $M\infty N$. However, consistency checks alone are not enough for the prescription. We have to elaborate further on general physical grounds.

Let Σ be a Riemann surface. We study a conformal field theory on it. A partition function Z_Σ is obtained as a functional integral over all possible configurations of the field variables on Σ. We can imagine that we cut Σ along loops C_1 and C_2 and obtain three surfaces, Σ_1, Σ_{12}, and Σ_2 (Figure 2). For fixed boundary conditions a_i and b_j on C_1 and C_2 we perform functional integrals on Σ_1, Σ_{12}, and Σ_2 separately. The resulting partition functions are $Z_{\Sigma_1}(a_i)$, $Z_{\Sigma_{12}}(a_i, b_j)$, and $Z_{\Sigma_2}(b_j)$. The Z_Σ is obtained as a product of the three summed over all possible boundary conditions,

$$Z_\Sigma=\sum_{i,j} Z_{\Sigma_1}(a_i)Z_{\Sigma_{12}}(a_i, b_j)Z_{\Sigma_2}(b_j) \tag{42}$$

This definition comes from general properties of the functional integral which say that the functional integral on Σ is equal to the product of the functional integrals on the cut surfaces with a suitable sum over a complete set of boundary conditions.

Now consider the inverse of the cutting. Suppose that we have two Riemann surfaces M and N (Figure 3) on which certain conformal fields

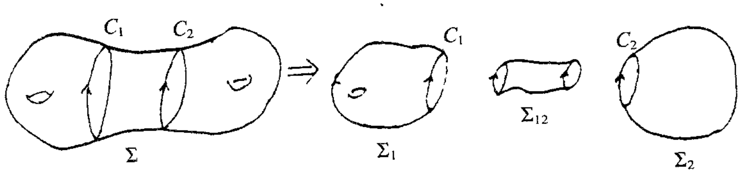


Fig. 2. Cutting of Σ along C_1 and C_2.

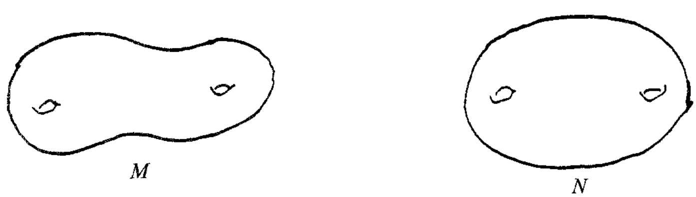
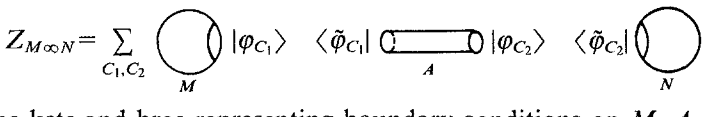


Fig. 3. Riemann surfaces M and N.

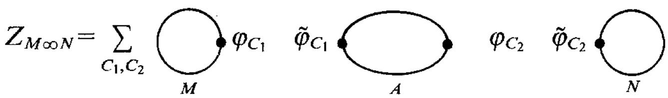
live with the same properties. The surfaces M and N can be sewn into a new surface $M \infty N$ according to the standard sewing construction of plumbing fixtures (Figure 4). By removing the interiors of the unit circles C_1 ($|z_1| = 1$) and C_2 ($|z_2| = 1$) and gluing in a cylinder A of length L, the surface $M \infty N$ is constructed.

Then the functional integral on $M \infty N$ can be separated into ones over M, A, and N with the fixed boundary conditions. For the case under consideration, equation (42) now can be diagrammed as follows:



(43)

with the kets and bras representing boundary conditions on M, A, and N. Using the state/field relation (Polchinski, 1988), we can also rewrite equation (43) as



(44)

For the conformal field under consideration we know exactly the complete set of local fields, and the local fields inserted on surfaces like φ_C etc., must be linear combinations of the primary fields and their descendants, namely of $\mathscr{L}_{-m}\varphi_i$. Moreover, in Hilbert space language, sewing the corresponding circles on the surfaces M, A, and N means identifying the corresponding Hilbert spaces. It is clear that the result of the functional integral on the surface $M \infty N$ is independent of how we specify the order for a pair of a ket and bra on the joining circle. As is well known, this requires a time-reversal operation in the Hilbert space, which is obtained by the appropriate choice of a real basis. In field language, we introduce conjugate fields to satisfy the requirement. Taking into account the above remark, we can obtain a standard form of the partition function in the standard notations

$$Z_{M\infty N} = \sum_{imn} \langle \cdots \mathscr{L}_{-m}\varphi_i(P)\rangle_M \langle \mathscr{L}_{-m}\varphi_i(w=0)\mathscr{L}_{-n}\tilde{\varphi}_j(z=0)\rangle_A \times \langle \mathscr{L}_{-n}\tilde{\varphi}_i(Q) \cdots \rangle_N \tag{45}$$

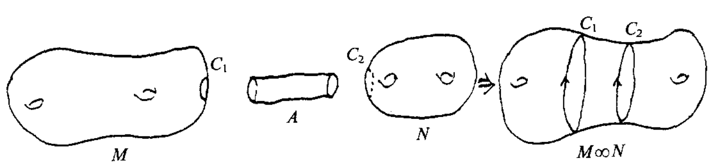


Fig. 4. Sewing of M, A and N, into $M \infty N$.

Under the choice of the metric on A

$$g_{z\bar{z}}=1/|z|^2 \tag{46}$$

the surface A is equivalent to a sphere with two punctures.

Note that the surfaces M and N are defined to have arbitrary topology and operator insertions, so equation (45) is a statement about correlation functions as well, namely,

$$\langle\theta_1\ldots\theta_k\theta_{K+1}\ldots\theta_{K+L}\rangle_{M\infty N}=\sum_{imn}\langle\theta_1\ldots\theta_k\mathscr{L}_{-m}\varphi_i(P)\rangle_M\mu_{imn}^{-1} \times\langle\mathscr{L}_{-n}\tilde{\varphi}_i(Q)\theta_{K+1}\ldots\theta_{K+L}\rangle_N \tag{47}$$

Equation (47) is what we want and in this way we complete the elaboration on the definition (30) in Section 4.

6. FINAL REMARKS

Conformal field theories provide a powerful tool with which to probe the structure of string theories. We have worked out formally the KN operator formalism, relying heavily on a sewing construction of conformal field theories and conformal techniques. It has been shown that if $M\infty N$ and $M'\infty N'$ are equivalent Riemann surfaces, the prescription gives the same theory (Sonoda, 1988*b*). However, a few questions remain open.

Generally speaking, the correlation functions of local fields on higher-genus Riemann surfaces should still satisfy systems of linear differential equations. What we need to do first is to develop a general method for deriving differential equations for the correlation functions on arbitrary Riemann surfaces except the sphere and torus (Belavin *et al.*, 1984; Mathur *et al.*, 1989). Then, we need to show how to derive differential equations for the correlation functions on $M\infty N$ from those for correlation functions on M and N, respectively.

We discuss these two questions elsewhere (Cheng, 1993).

Also, it would be of interest to construct KN operator formalisms for other conformal field theories on higher-genus Riemann surfaces. It is expected that this extension is not very difficult.

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